Background models in Least Squares Adjustment (LSA)

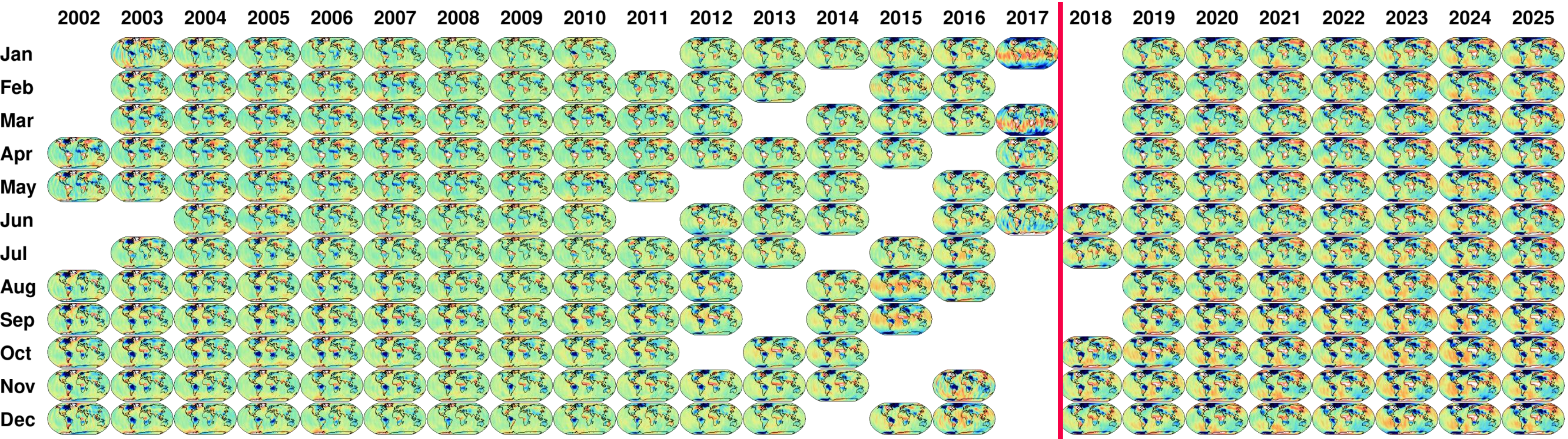
Torsten Mayer-Gürr

Institute of Geodesy
Graz University of Technology

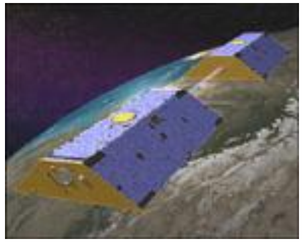
UAW 2026, Munich

2026-03-05

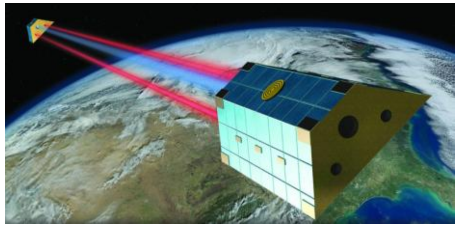
ITSG-Grace monthly gravity field solutions



GRACE



GRACE-FO



ITSG-Grace2018: Monthly and Daily solutions
ifg.tugraz.at/downloads/gravity-field-models/itsg-grace2018/

Least squares adjustment

Remove – Compute – Restore:

- Observation equations

$$\mathbf{l} = \mathbf{f}(\mathbf{x}) + \mathbf{e}$$

- Linearization by a truncated Taylor series

$$\mathbf{l} = \mathbf{f}(\mathbf{x}_0) + \left.\frac{\partial f}{\partial \mathbf{x}}\right|_0 (\mathbf{x} - \mathbf{x}_0) + \mathbf{e}$$

- „Remove“: observed minus computed

$$\Delta\mathbf{l} = \mathbf{l} - \mathbf{f}(\mathbf{x}_0)$$

- „Compute“

$$\Delta\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\Delta\mathbf{l}$$

- „Restore“

$$\hat{\mathbf{x}} = \mathbf{x}_0 + \Delta\hat{\mathbf{x}}$$

- Very general: GRACE, GNSS, VLBI, SLR, constructing ITRF
- => Think about a LSA that you're familiar with.

Least squares adjustment

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

Complex Earth model

- Solving a system of differential equations
- More data/parameters than observations
  - Atmosphere, ocean, hydrology
  - Solid Earth tides
  - Sun and moon ephemeris
  - Earth rotation
  - …

Least squares adjustment

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

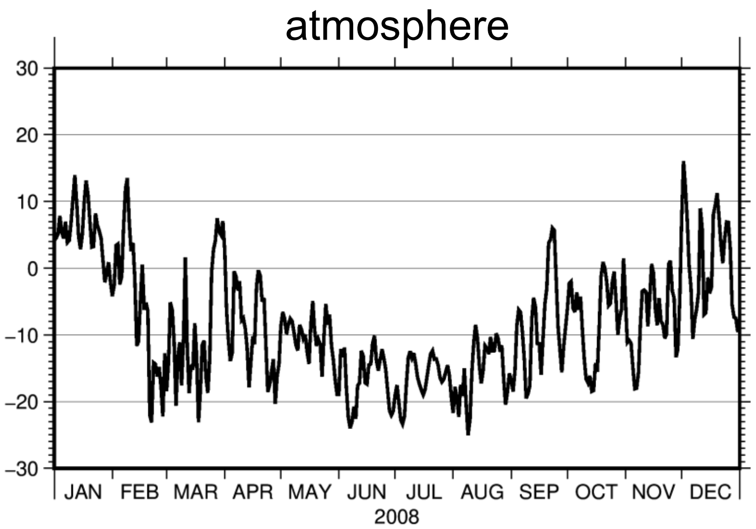
Least squares adjustment

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

Adjustable parameters
- Monthly gravity
- …

All other fixed data/parameters
- Submonthly temporal gravity
- Ephemeris of sun and moon
- …



Least squares adjustment

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

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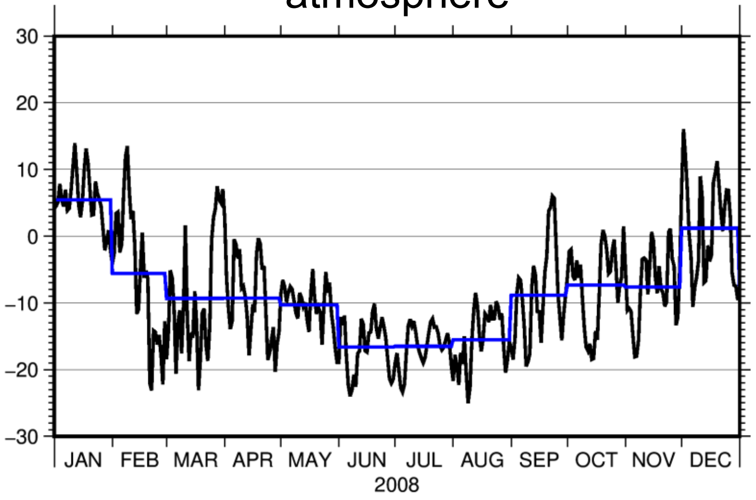
atmosphere



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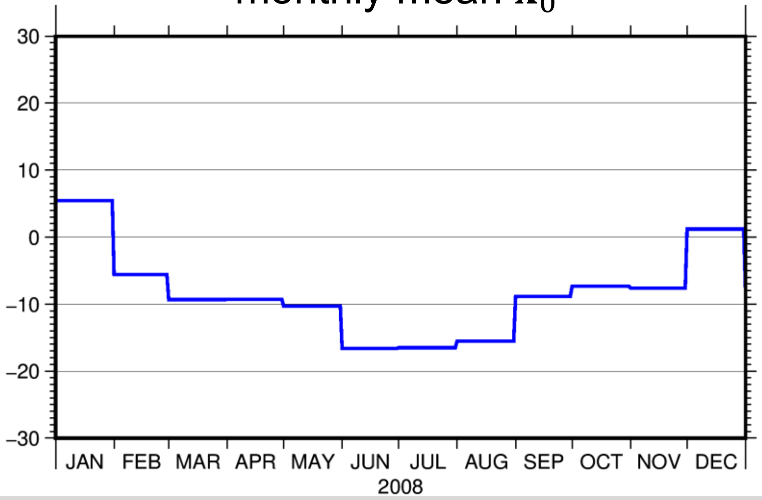
monthly mean $\mathbf{x}_0$



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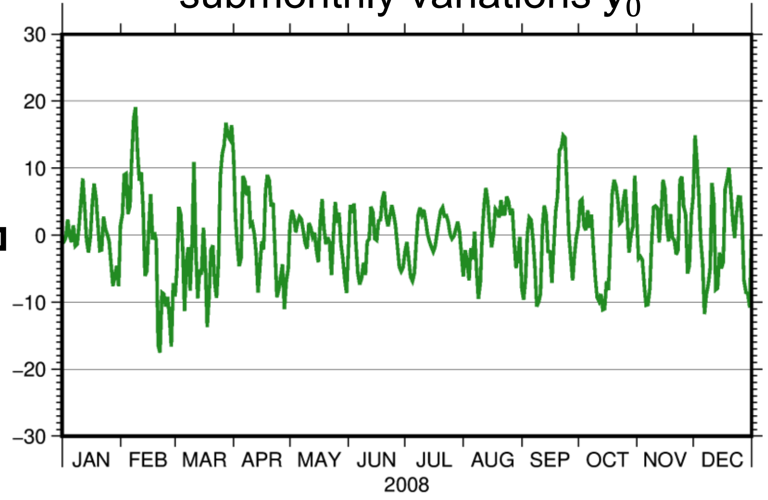
submonthly variations $\mathbf{y}_0$



Least squares adjustment

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

Adjustable parameters
- Monthly gravity
- …

All other fixed data/parameters
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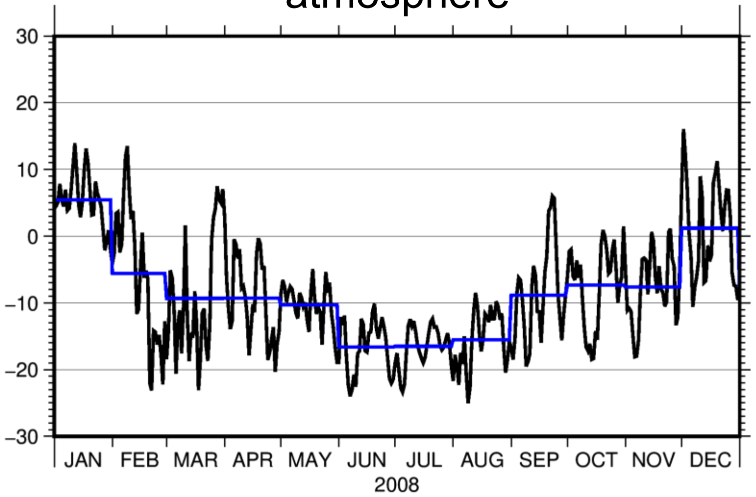
monthly mean $\mathbf{x}_0$

- $\mathbf{x}_0$ causes no aliasing (it is constant in the retrieval perdiod)
- Important for restore: $\hat{\mathbf{x}} = \mathbf{x}_0 + \Delta\hat{\mathbf{x}}$ (product definition)
- Model accuracy of $\mathbf{x}_0$ does not play a role

atmosphere



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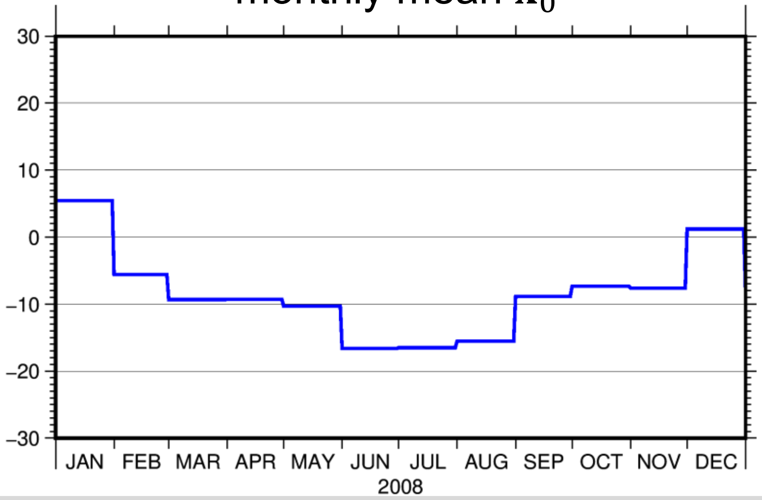

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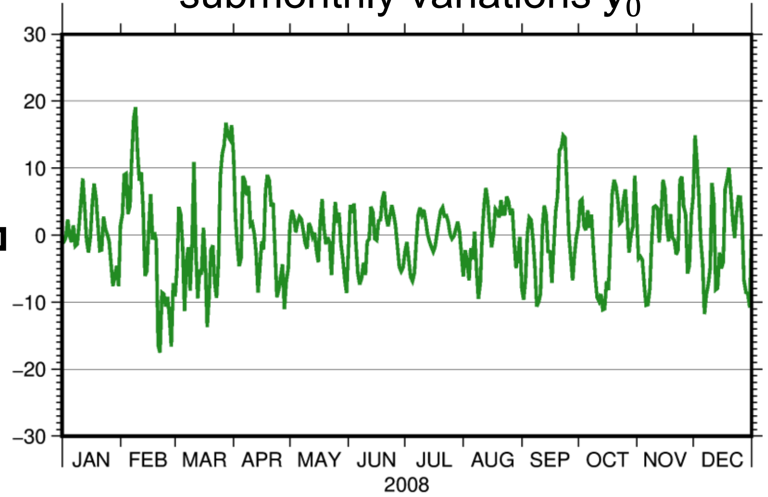

Least squares adjustment

- Observation equations

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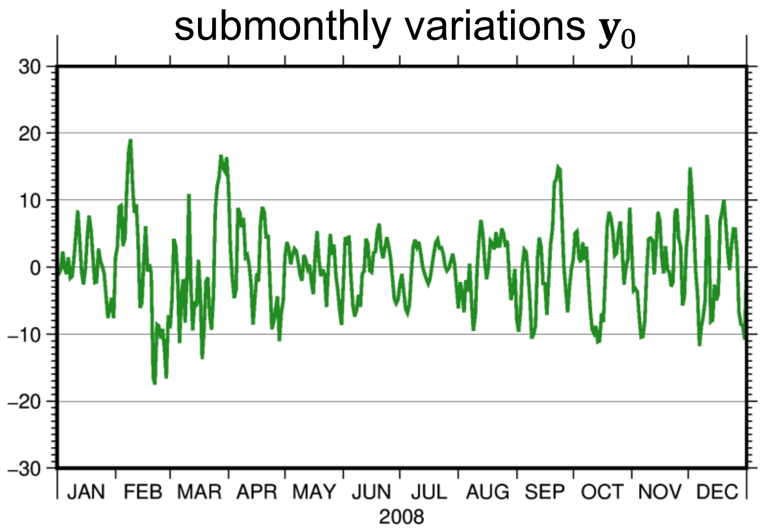
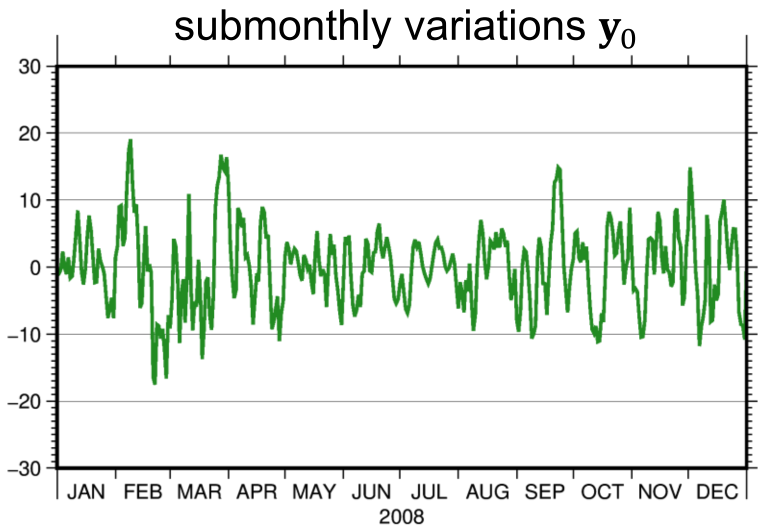


Stochastic Modelling

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

- Models (as observations) are not perfect
- Uncertainty can be described by a spatial-temporal covariance matrix $\mathbf{\Sigma}(\mathbf{y}_0)$
- E.g. GFZ provides simulated error times series derived from an ensemble of model runs (atmosphere, ocean, hydrology)



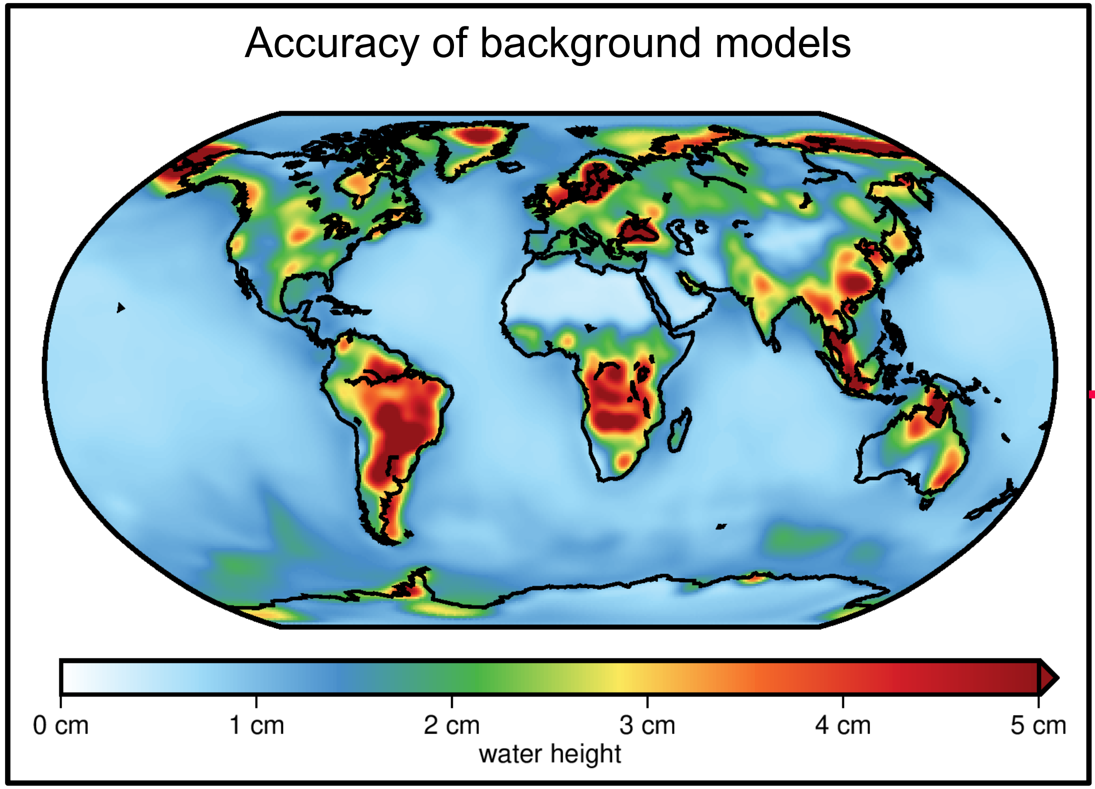
Stochastic Modelling

- Observation equations

$$\mathbf{l} - \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{A}\Delta\mathbf{x} + \mathbf{e} \qquad \text{with} \qquad \mathbf{\Sigma}(\Delta\mathbf{l}) = \mathbf{\Sigma}(\mathbf{l}) + \mathbf{B}\mathbf{\Sigma}(\mathbf{y}_0)\mathbf{B}^T$$

Reduced observations

observations



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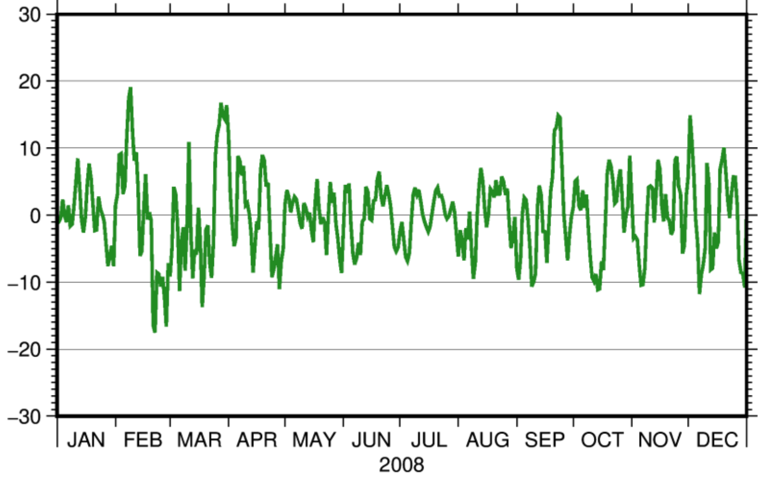
submonthly variations $\mathbf{y}_0$



Stochastic Modelling

- Observation equations

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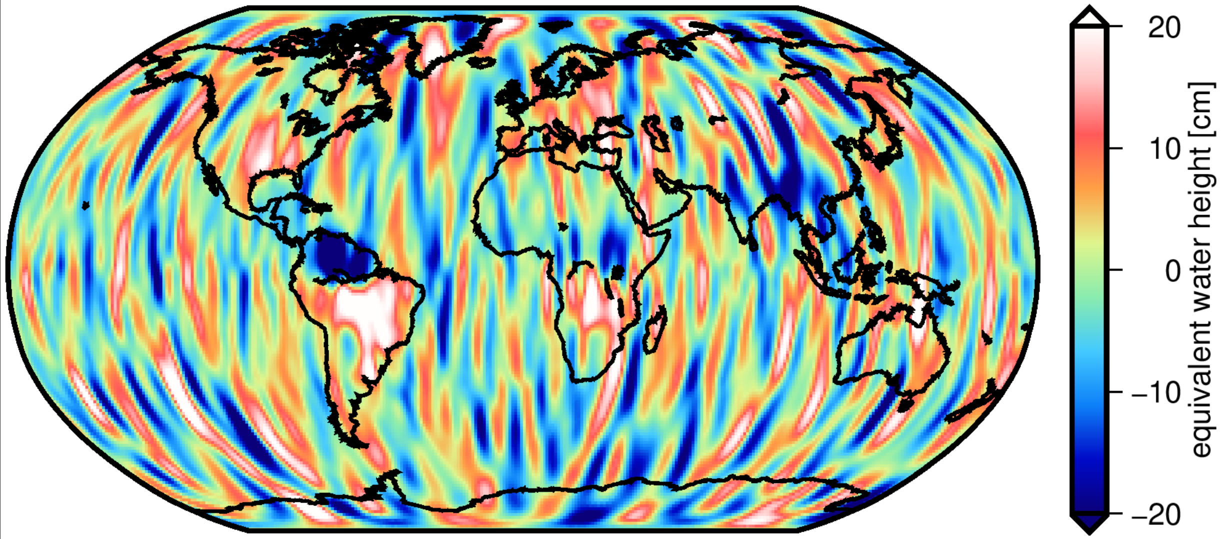
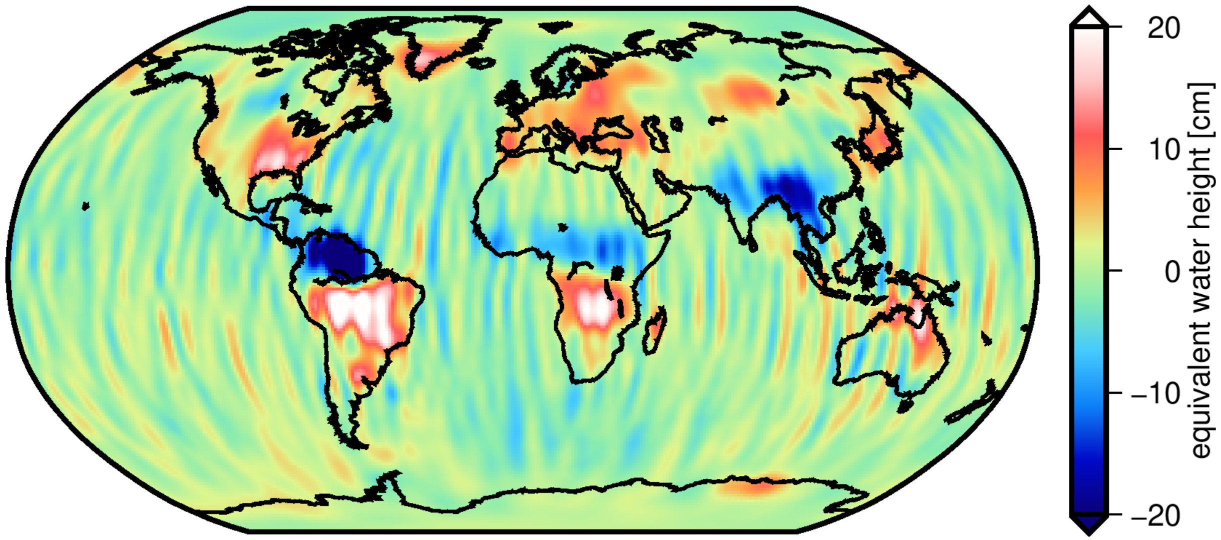
Solution without **diagonal** covariance matrix

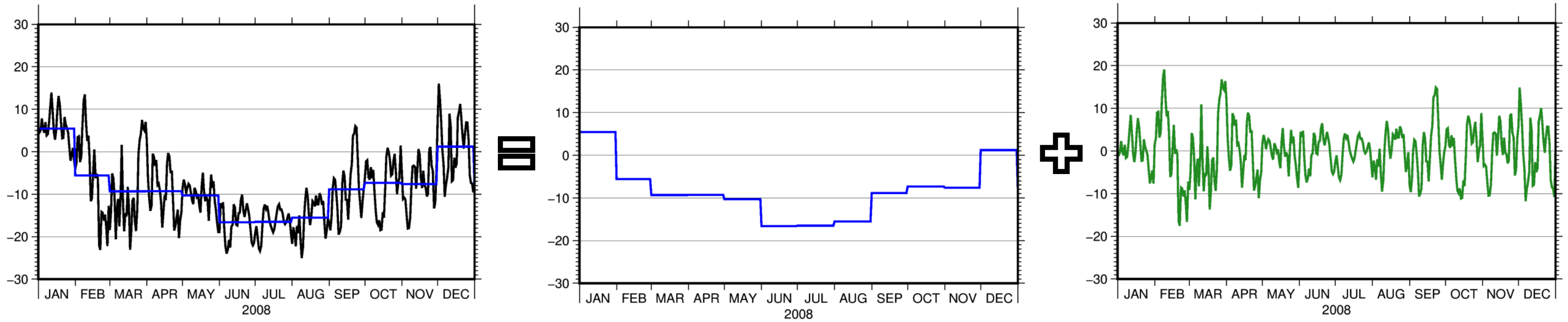


Solution with **tailored** covariance matrix



Summary

- Please distinguish between $\mathbf{x}_0$ and $\boldsymbol{y}_0$ in the discussion about the use of models.



- My personal opinion: Geodetic products should contain the full signal (adding back the model: $\hat{\mathbf{x}} = \mathbf{x}_0 + \Delta\hat{\mathbf{x}}$)
  - Independent of models
    - Gravity fields
    - Station coordinate time series
    - Earth Orientation parameters

- Models (as observations) contain uncertainties
  - Uncertainties may be described by a covariance matrix $\boldsymbol{\Sigma}(\mathbf{y}_0)$